MTH 295
Fall 2019
Homework 1
Due Thursday, 9/12

Name: Key

1) For each of the following ODEs, determine its order and degree and state whether it is linear or non-linear.

a) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \sin x$

2nd order, 1st degree, linear

b) $(1+y^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = e^x$

2nd order, 1st degree, non-linear

c) $(\dot{x})^3 + tx^2 = 0$

1st order, 3rd degree, non-linear

d) $u' + uu' = 1 + u''$

$\boxed{\text{2nd order, 1st degree, non-linear}}$

e) $y'' + \sin(x+y) = \cos x$

$\boxed{\text{2nd order, 1st degree, non-linear}}$

f) $e^x f''(x) - 3\tanh(x) f'(x) = 6 f(x)$

$\boxed{\text{2nd order; 1st degree, linear}}$

2) For each of the following equations, determine all values of k for which the equation has a solution of the given form.

a) $\ddot{x}+4\dot{x}-x=0$, $x=e^{kt}$

If $x=e^{kt}$
then $\dot{x}=ke^{kt}$
$\ddot{x}=k^2e^{kt}$

If x satisfies the equation then
$k^2e^{kt}+4ke^{kt}-e^{kt}=0$

$k^2+4k-1=0$

$k=\dfrac{-4\pm\sqrt{16+4}}{2}$

$\boxed{k=-2\pm\sqrt{5}}$

b) $x^2y''-4xy'+4y=0$, $y=x^k$

$y=x^k$
$y'=kx^{k-1}$
$y''=k(k-1)x^{k-2}$

so $x^2\cdot k(k-1)x^{k-2}-4x\,kx^{k-1}+4x^k=0$

$k(k-1)-4k+4=0$

$k^2-5k+4=0$

$(k-4)(k-1)=0$

$\boxed{k=4 \text{ or } k=1}$

3) a) Show that $y = c_1e^{2x} + c_2e^{-3x} + xe^{2x}$ is a solution of the second order ODE $y'' + y' = 6y + 5e^{2x}$.

$y = c_1e^{2x} + c_2e^{-3x} + xe^{2x}$

$y' = 2c_1e^{2x} - 3c_2e^{-3x} + e^{2x} + 2xe^{2x}$

$y'' = 4c_1e^{2x} + 9c_2e^{-3x} + 2e^{2x} + 2e^{2x} + 4xe^{2x}$

so $y'' + y' = 4c_1e^{2x} + 9c_2e^{-3x} + 4e^{2x} + 4xe^{2x} + 2c_1e^{2x} - 3c_2e^{-3x} + e^{2x} + 2xe^{2x}$

$= 6c_1e^{2x} + 6c_2e^{-3x} + 6xe^{2x} + 5e^{2x}$

$= 6(c_1e^{2x} + c_2e^{-3x} + xe^{2x}) + 5e^{2x}$

$= 6y + 5e^{2x}$ ✓

so y is a solution of given equation.

b) Find the particular solution to the IVP consisting of the ODE above with the two initial conditions $y(0) = y'(0) = 1$.

From above –

$y(0) = c_1 + c_2 = 1$

$y'(0) = 2c_1 - 3c_2 + 1 = 1$

$c_1 + c_2 = 1$

$2c_1 - 3c_2 = 0$

$5c_2 = 2$

$c_2 = \frac{2}{5}$

$c_1 = 1 - c_2 = 1 - \frac{2}{5} = \frac{3}{5}$

so $\boxed{y = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x} + xe^{2x}}$ is the particular solution

4) Solve the IVP $\ddot{x} = \cos(\omega t), x(0) = \dot{x}(0) = 0$ where ω is an arbitrary constant.

$\ddot{x} = \cos(wt)$

$\dot{x} = \int \cos(wt)dt = \frac{1}{w}\sin(wt) + c_1$

$x = \int \dot{x}\,dt = -\frac{1}{w^2}\cos(wt) + c_1 t + c_2$

apply conditions –

$x(0) = -\frac{1}{w^2} + c_2 = 0$

$c_2 = \frac{1}{w^2}$

$\dot{x}(0) = c_1 = 0$

so $\boxed{x = -\frac{1}{w^2}\cos(wt) + \frac{1}{w^2}}$

5) Show that if L is a linear operator, i.e. $L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$, then $L(c_1 f_1 + c_2 f_2 + c_3 f_3) = c_1 L(f_1) + c_2 L(f_2) + c_3 L(f_3)$. Hint: you might find it easier to begin with the right hand side of this expression.

You don't have to prove this but it should be obvious after you're done that this extends inductively to a linear combination of any number of functions.

$c_1 L(f_1) + c_2 L(f_2) + c_3 L(f_3) = L(c_1 f_1 + c_2 f_2) + c_3 L(f_3)$

$= (1) L(c_1 f_1 + c_2 f_2) + c_3 L(f_3)$

$= L((1)(c_1 f_1 + c_2 f_2) + c_3 f_3)$

$= L(c_1 f_1 + c_2 f_2 + c_3 f_3)$ //